2019

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following as directed : 1×7=7

(a) If $V = R^n(R)$, then
$W = \{(v_1, v_2, ..., v_n) : v_1 + v_2 + ... + v_n = 1\}$
is a subspace of V.

(Disprove it)

(b) If $v_i = v_j$ for some $i \neq j$, then the sequence $v_1, v_2, \ldots, v_n$ of vectors in a vector space is

(i) linearly independent

(ii) linearly dependent

(Choose the correct option)

(c) Consider the complex field C which contains the real field R. Show that $\{1, i\}$ is a basis of the vector space C over R, where $i = \sqrt{-1}$.

(d) Suppose $T : R^5 \to R^2$ is a linear transformation defined by $T(x) = Ax$ for some matrix A and for each x in R^5. How many rows and columns does A have?

(e) For a linear operator (matrix) T, the scalar 0 is an eigenvalue of T if and only if T is singular.

(Write True or False)

(f) Find the minimal polynomial $m(t)$ of the following matrix :

$$M = \begin{bmatrix} -5 & 4 \\ 2 & 9 \end{bmatrix}$$

(g) If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $x \mapsto Ax$ is a linear transformation on C^2, where C is the complex field. Show that $v = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector of A.

2. Answer the following questions : 2×4=8

(a) Determine if the columns of the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

are linearly independent.

(b) Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and define $T: M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that T is a linear transformation.

(c) Let $a_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $b_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $b_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the bases for R^2 given by $E = \{a_1, a_2\}$ and $F = \{b_1, b_2\}$. Find the change-of-coordinate matrix from E to F.

(d) If λ is an eigenvalue of a linear operator $T: V \to V$, then prove that the set E_λ of all eigenvectors belonging to λ is a subspace of V.

3. Answer any *one* part : 5

(a) Let V_1 and V_2 be vector spaces over the same field F. For any linear transformation $T: V_1 \to V_2$, prove that $r(T) \le \min(\dim V_1, \dim V_2)$, where $r(T)$ denotes the rank of T.

(b) Define the dual space V^* of a vector space V and prove that if V is the finite-dimensional vector space over a field F, then for any $u(\neq 0)$ in V there exists $g \in V^*$ such that $g(u) \neq 0$.

4. Answer the following questions :

(a) If $u_1, u_2, \ldots, u_n$ are non-zero linearly dependent vectors in a vector space V over a field F, then prove that for some i, $2 \le i \le n$, u_i is a linear combination of its predecessors $u_1, u_2, \ldots, u_{i-1}$ and the subspace spanned by $\{u_1, u_2, \ldots, u_n\}$ is same as that spanned by $\{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n\}$. 10

Or

Suppose V is a finite dimensional vector space over a field F and U is a subspace of V. Prove that there is a subspace W of V such that $V = U \oplus W$. 10

(b) (i) If a linear transformation $T : V \to V$ has n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, then prove that V has an ordered basis $\{u_1, u_2, ..., u_n\}$ such that the matrix of T related to this basis is

$$\begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda_n \end{bmatrix}$$

(ii) State the existence and uniqueness theorem of solution of a system of linear equations. Determine the existence and uniqueness of the solution of the system whose augmented matrix after row reduced is

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

5+5=10

Or

What do we mean by the minimal polynomial of a matrix (linear operator)? Let T be the operator on R^2 which projects each vector onto the x-axis, parallel to the y-axis :

$$T(x, y) = (x, 0)$$

Show that T is linear. What is the minimal polynomial for T? 10

GROUP—B

(Vector)

(*Marks* : 40)

5. Answer the following questions : 1×3=3

(a) Prove that $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$.

(b) If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then define the reciprocal vector of $\vec{a}$.

(c) If the four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar, then show that

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$$

6. Find the constant λ such that the following vectors are coplanar : 2

$$\vec{a} = 2\vec{i} - \vec{j} + \vec{k},\ \vec{b} = \vec{i} + 2\vec{j} - 3\vec{k},$$

$$\vec{c} = 3\vec{i} + \lambda\vec{j} + 5\vec{k}$$

7. Answer the following questions :

(a) Prove that for any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$. 5

(b) Prove that the necessary and sufficient condition for a vector $\vec{v}(t)$ to have a constant direction is

$$\vec{v} \times \frac{d\vec{v}}{dt} = \vec{0}$$ 5

(c) Prove that

$$\text{curl}\,(\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - \vec{B}\ \text{div}\,\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}\ \text{div}\,\vec{B}$$ 5

Or

Taking $\vec{f} = x^2 y\vec{i} + xz\vec{j} + 2yz\vec{k}$, verify that $\text{div}\,(\text{curl}\,\vec{f}) = 0$. 5

8. Answer the following questions :

(a) *(i)* If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$, then find

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| \text{ and } \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) \cdot \frac{d^3\vec{r}}{dt^3}$$ 2+3=5

(ii) A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$, where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction $\vec{i} + \vec{j} + 3\vec{k}$. 5

Or

(i) If $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$, then find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, where $\vec{b}$ and $\vec{c}$ being the parallel. 4

(ii) If $\frac{d\vec{u}}{dt} = \vec{w} \times \vec{u}$, $\frac{d\vec{v}}{dt} = \vec{w} \times \vec{v}$, then show that $\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{w} \times (\vec{u} \times \vec{v})$. 3

(iii) Prove that div $\vec{r} = 3$. 3

(b) If $\vec{F} = y\vec{i} - x\vec{j}$, then evaluate $\int_C \vec{F} \cdot d\vec{r}$ from (0, 0) to (1, 1) along the following paths C:

(i) The parabola $y = x^2$

(ii) The straight lines from (0, 0) to (1, 0) and then to (1, 1)

(iii) The straight line joining (0, 0) and (1, 1). 4+3+3=10

Or

If $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$ and S is the surface of the sphere

$$x^2 + y^2 + z^2 = a^2,\ 0 \le x, y, z \le a$$

then evaluate $\iint_S \vec{F} \cdot \vec{n}\, d\vec{S}$, where $\vec{n}$ is a unit vector along the outward down normal to the sphere S. 10

★ ★ ★