Total number of printed pages–28

3 (Sem–6/CBCS) MAT HE 1/2/3/4

2023

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION – A

(Boolean Algebra and Automata Theory)

Paper : MAT–HE–6016

Full Marks : 80

Time : Three hours

OPTION – B

(Biomathematics)

Paper : MAT–HE–6026

Full Marks : 80

Time : Three hours

OPTION – C

(Mathematical Modeling)

Paper : MAT–HE–6036

Full Marks : 60

Time : Three hours

OPTION – D

(Hydromechanics)

Paper : MAT–HE–6046

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

OPTION–A

(Boolean Algebra and Automata Theory)

Paper : MAT–HE–6016

1. Answer the following questions : 1×10=10

   (a) A relation $\leq$ on a set P is called quasi-order, if

   (i) reflexive, transitive and antisymmetric

   (ii) reflexive and antisymmetric

   (iii) transitive and antisymmetric

   (iv) None of the above

   (Choose the correct answer)

   (b) An ordered set P is an antichain if ________ in P only if ________.

   (Fill in the blanks)

   (c) Let P^D be the dual of any ordered set P. Then

   (i) $x \leq y$ holds in P^D if $x \leq y$ holds in P

   (ii) $x \leq y$ holds in P^D if $y \leq x$ holds in P

   (iii) $x \leq y$ holds in P^D if $x = y$ holds in P

   (iv) None of the above

   (Choose the correct answer)

(d) Define lattice homomorphism.

(e) Let L be a lattice and $a, b \in L$. If $a \leq b$, then

(i) $a \vee b = b, a \wedge b = a$

(ii) $a \vee b = b$ but not $a \wedge b = a$

(iii) $a \wedge b = a$ but not $a \vee b = b$

(iv) None of the above

(Choose the correct answer)

(f) Define conjunctive normal form.

(g) For all x, y in a Boolean algebra,

(i) $(x \wedge y)' = x' \vee y'$ and $(x \vee y)' = x' \wedge y'$

(ii) $(x \wedge y)' = x' \wedge y'$ and $(x \vee y)' = x' \vee y'$

(iii) $(x \wedge y)' = y'$ and $(x \vee y)' = x'$

(iv) None of the above

(Choose the correct answer)

(h) Define Boolean polynomial function.

(i) What is the empty string?

(j) Define closure properties of regular languages.

2. Answer the following questions : 2×5=10

(a) Prove that the elements of any arbitrary lattice satisfy the following inequalities :

(i) $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$

(ii) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$

(b) Prove that every chain is a distributive lattice.

(c) Define NFA.

(d) Define atom. Prove that every atom of a lattice with zero is join-irreducible.

(e) Prove that if L and M are regular languages, then $L \cup M$ is also a regular language.

3. Answer ***any four*** questions from the following : 5×4=20

(a) (i) Prove that two finite ordered set P and Q are order-isomorphic if and only if they can be drawn with identical diagrams.

(ii) Define monomorphism. Let f be a monomorphism from the lattice L into the lattice M. Show that L is isomorphic to a sublattice M.

(b) (i) Let C_1 and C_2 be the finite chains $\{0, 1, 2\}$ and $\{0, 1\}$ respectively. Draw the Hasse diagram of the product lattice $C_1 \times C_2 \times C_3$.

(ii) Let L be a distributive lattice with 0 and 1. Prove that if a has a complement a', then

$$a \vee (a' \wedge b) = a \vee b.$$

(c) (i) State and prove De Morgan's laws of a Boolean algebra.

(ii) Let $f: B_1 \to B_2$ be a Boolean homomorphism. Then prove the following :

(1) $f(0) = 0,\ f(1) = 1$

(2) For all $x, y \in B_1$

$$x \le y \Rightarrow f(x) \le f(y).$$

(d) Let $p, q \in P_n$; $p \sim q$ and let B be an arbitrary Boolean algebra. Then, prove that $\overline{p}_B = \overline{q}_B$.

(e) Prove that a language L is accepted by some DFA if and only if L is accepted by some NFA.

(f) Prove that every regular language is a context-free language.

4. Answer the following questions : 10×4=40

(a) (i) Let P and Q be finite ordered sets and let $f : P \to Q$ be a bijective map. Then, prove that the following are equivalent :

(1) f is an order-isomorphism;

(2) $x < y$ in P if and only if $f(x) < f(y)$ in Q;

(3) $x \prec y$ in P if and only if $f(x) \prec f(y)$ in Q. 5

(ii) Let P be an ordered set. Then, prove that

$O(P \oplus 1) \cong O(P) \oplus 1$ and

$O(1 \oplus P) \cong \oplus 1\, O(P)$ 5

OR

Let P be a finite ordered set.

(i) Show that $Q = \downarrow Max\, Q$, for all $Q \in O(P)$

(ii) Establish a one-to-one correspondence between the elements of $O(P)$ and antichains in P

(iii) Hence show that for all $x \in P$,

$$|O(P)| = |O(P \setminus \{x\})| + |O(P \setminus (\downarrow x U \uparrow x))|$$

10

(b) *(i)* Let L be a distributive lattice and let $P \in L$ be join-irreducible with $p \leq a \vee b$. Then, prove that $p \leq a$ or $p \leq b$. 5

(ii) Prove that generalized distributive inequality for lattices

$$y \wedge \left(\overset{n}{\underset{i=1}{\vee}} x_i \right) \geq \overset{n}{\underset{i=1}{\vee}} (y \wedge x_i).$$ 5

OR

(iii) Let B be a Boolean algebra. Then, prove that the set $P_n(B)$ is a Boolean algebra and subalgebra of the Boolean algebra $F_n(B)$ of all functions from B_n into B. 5

(iv) Find the DNF of

$$x_1(x_2+x_3)' + (x_1x_2+x_3')x_1$$ 5

(c) *(i)* Prove that a polynomial $p \in P_n$ is equivalent to the sum of all prime implications of p. 5

(ii) Find three prime implications of $xy + xy'z + x'y'z$. 5

OR

(iii) Determine the symbolic representation of the circuit given by

$$p = (x_1+x_2+x_3)(x_1'+x_2)(x_1x_3+x_1'x_2)(x_2'+x_3)$$

5

(iv) Design a switching circuit that enables you to operate one lamp in a room from four different switches in that room. 5

(d) *(i)* If L, M and N are any languages, then prove that

$$L(M \cup N) = LM \cup LN.$$ 5

(ii) If L is a regular language over alphabet Σ, then $\overline{L} = \Sigma^* - L$ is also a regular language. 5

OR

(iii) Consider the CFG K defined by productions

$$S \to aSbS \mid bSaS \mid \varepsilon$$

Prove that $L(K)$ is the set of all strings with an equal number of a's and b's. 5

(iv) Let $G = (V, T, P, S)$ be a CFG, and suppose that there is a derivation

$A \overset{*}{\underset{G}{\Rightarrow}} w$, where w is in T^*. Then, prove that the recursive inference procedure applied to G determines that w is in the language of variable A. 5

OPTION–B

(Biomathematics)

Paper : MAT–HE–6026

1. Answer the following questions : 1×10=10

(a) What is an autonomous system?

(b) The zero equilibrium/positive equilibrium is often not a desired state in biological system.

(Choose the correct answer)

(c) Write a difference between continuous growth and discrete growth.

(d) Give an example of nonlinear, autonomous second order difference equation.

(e) Write *one* use of Routh-Hurwitz criteria.

(f) Equilibria are also known as

(a) steady state

(b) fixed points

(c) critical points

(d) All of the above

(Choose the correct answer)

(g) Write the condition that a first order partial derivative of a system is locally asymptotically stable.

(h) Write the condition that the equilibrium $\bar{x}$ of $\frac{dx}{dt} = f(x)$ is hyperbolic.

(i) Write the three population classes in Kermack-McKendrick model.

(j) Define a characteristic polynomial for second order equation.

2. Answer the following questions : 2×5=10

(a) Define a difference equation of order k.

(b) State Frobenius theorem.

(c) Distinguish between local stability and global stability.

(d) Consider the linear differential equation

$$\frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + \frac{dx}{dt} + ax = 0$$

Show that its solution approaches zero.

(e) For the linear differential equation $\frac{dx}{dt} = AX$, the matrix A is given by $A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$. Find the eigenvalues.

3. Answer ***any four*** questions : 5×4=20

(a) The difference equation is given by $x_{t+4} + ax_t = 0$.
Find its characteristic equation and its solutions.

(b) Find the eigenvalues and eigenvectors of matrix A when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Then find the general solutions to $x(t+1) = Ax(t)$.

(c) Find all the equilibria for the difference equation $x_{t+1} = ax_t \; exp\,(-rx_t),\ a, r > 0$.

(d) Consider the differential equation

$$x'''(t) - 4x''(t) = 0$$

where $x'' = \frac{d^2x}{dt^2}$ and so on.

Find its characteristic equation and its roots or eigenvalues and verify that the solutions are linearly independent or not.

(e) A mathematical model for the growth of a population is

$$\frac{dx}{dt} = \frac{2x^2}{1+x^4} - x = f(x),\ x(0) \geq 0$$

where x is the population density. Find the equilibria and determine their stability.

(f) Suppose an SIS epidemic model with disease-related deaths and a growing population satisfies

$$\frac{dN}{dt} = N(b - CN) - \alpha I,\ b,\ c,\ \alpha > 0$$

(i) Find the differential equations satisfied by the proportions

$$i(t) = \frac{I(t)}{N(t)} \text{ and } s(t) = \frac{S(t)}{N(t)}$$

Then find the basic reproduction number.

(ii) Do the dynamics of $N(t)$ change with disease? Is it possible for $N(t) \to 0$? Note that $m(N) = CN$ and $\frac{dN}{dt} = N(b - CN - \alpha i)$.

4. Answer the following questions : 10×4=40

(a) Find the general solution to the non-homogeneous linear difference equation $x_{t+2} + x_{t+1} = 6x_t = 5$

Or

Suppose the Leslie matrix is given by

$$L = \begin{pmatrix} 0 & \frac{3a^2}{2} & \frac{3a^3}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}, \ a > 0$$

(i) Find the characteristic equation, eigenvalues and inherent net reproduction number R_0 of L.

(ii) Show that L is primitive.

(iii) Find the stable age distribution.

(b) The following epidemic model is referred to as an SIS epidemic model. Infected individuals recover but do not become immune. They become immediately susceptible again.

$$S_{t+1} = S_t - \frac{\beta}{N} I_t S_t + (\gamma + b) I_t$$

$$I_{t+1} = I_t (1 - \gamma - b) + \frac{\beta}{N} I_t S_t$$

Assume that $0 < \beta < 1$, $0 < b + \gamma < 1$. $S_0 + I_0 = N$ and $S_0, I_0 > 0$

(i) Show that $S_t + I_t = N$ for $t = 1, 2, \ldots$

(ii) Show that there exist two equilibria and they are both non-negative if $R_0 = \frac{\beta}{b + \gamma} \gtrless 1$.

Or

Discuss a predator-prey model with a suitable example by finding its equilibria, local stability and global stability.

(c) State briefly a measles model with vaccination.

Or

Show that the solution to the pharmacokinetics model is

$$x(t) = \frac{1}{a}\left(1 - e^{-at}\right)$$

$$y(t) = \frac{1}{b} + \frac{e^{-at}}{a-b} - \frac{ae^{-bt}}{b(a-b)}$$

(d) For the following differential equation, find the equilibria, then graph the phaseline diagram. Use the phaseline diagram to determine the stability of equilibrium

$$\frac{dx}{dt} = x(a-x)(x-b)^2, \; 0 < a < b.$$

Or

Discuss briefly about simple Kermack-McKendric epidemic model.

OPTION–C

(Mathematical Modeling)

Paper : MAT–HE–6036

1. Answer the following questions : $1\times7=7$

   (a) Write Legendre's equation of order n.

   (b) When does a power series converge if f be the radius of convergence and $0<\rho<\infty$?

   (c) Write the value of $\Gamma 3$.

   (d) Find the Laplace transform of $F(t)=1$.

   (e) Monte Carlo simulation is a probabilistic/logistic model.
   (Choose the correct answer)

   (f) The linear congruence method was introduced by ________.
   (Fill in the blank)

(g) Which one is not a high level simulation language?

(i) GPSS

(ii) SPSS

(iii) SIMAN

(iv) DYNAMO

(Choose the correct answer)

2. Answer the following questions : 2×4=8

(a) Show that $\sqrt{x+1} = x\sqrt{x}$.

(b) Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s-3)}.$$

(c) Write *two* advantages of Monte Carlo simulation.

(d) Why is sensitivity analysis important in linear programming?

3. Answer ***any three*** questions of the following : 5×3=15

(a) Solve the equation

$$y' + 2y = 0$$

(b) Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(1+x)y'' + 3x(1+x)^3 y' - (1-x^2)y = 0$$

(c) Suppose that m is a positive integer. Show that

$$\Gamma\left(m+\tfrac{2}{3}\right) = \frac{2.5.8....(3m-1)}{3^m}\,\Gamma\left(\tfrac{2}{3}\right).$$

(d) Solve the equation

$$4x^2y'' + 8xy' + (x^4 - 3)y = 0$$

(e) Write briefly about different steps of the simplex method.

4. Answer the following : 10×3=30

(a) Solve the initial value problem

$$\left(t^2 - 2t - 3\right)\frac{d^2y}{dt^2} + 3(t-1)\frac{dy}{dt} + y = 0;$$

$y(1) = 4, \ y'(1) = -1$

Or

Find the Frobenius series solutions of $xy'' + 2y' + xy = 0$.

(b) Using Monte Carlo simulation, write an algorithm to calculate an approximation to π by considering the number of random points selected inside the quarter circle.

$$Q : x^2 + y^2 = 1, \ x \geq 0, y \geq 0$$

where the quarter circle is taken to be inside the square

$S : 0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Or

Solve the equation $y'' + y = 0$.

(c) Write briefly about middle square method.

Or

A small harbor has unloading facilities for ships. Only one ship can be unloaded at any time. The unloading time required for a ship depends on the type and amount of cargo and varies from 45 to 90 minutes.

Below is given a situation with 5 ships :

	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5
Time between successive ships : (in minutes)	30	15	20	25	120
Unloading time :	40	35	60	45	75

(i) Draw the time line diagram depicting clearly the situation for each ship, the idle time for the harbor and the waiting time.

(ii) List the waiting time for all the ships and find the average waiting time.

OPTION–D

(Hydromechanics)

Paper : MAT–HE–6046

1. Answer the following questions : 1×10=10

   (a) What happens when there is an increase of pressure at any point of a liquid at rest under given external forces?

   (b) State Charles' law.

   (c) What is internal energy?

   (d) Define adiabatic expansion.

   (e) Give an example of application of atmospheric pressure in daily life.

   (f) Define ideal fluid.

   (g) Potential flow is the .... flow of an inviscid or perfect flow.

   (Fill in the gap)

(h) Equation of continuity by Euler's method is

(i) $\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{a} = 0$

(ii) $\frac{\partial \rho}{\partial t} - \rho \nabla \cdot \vec{a} = 0$

(iii) $\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \vec{a}) = 0$

(iv) None of the above

(Choose the correct option)

(i) Streamlines and pathlines become the same when the motion is ....

(Fill in the gap)

(j) Velocity potential φ satisfies which of the following equations?

(i) Bernoulli

(ii) Cauchy

(iii) Laplace

(iv) None of the above

(Choose the correct option)

2. Answer the following questions : 2×5=10

(a) Show that the surfaces of equal pressure are intersected orthogonally by the lines of force.

(b) Define field of force and line of force with examples.

(c) If ρ_0 and ρ be the densities of a gas at 0° and $t°$ Centigrade respectively, then establish the relation $\rho_0 = \rho(1 + \alpha t)$ where $\alpha = \dfrac{1}{273}$.

(d) Distinguish between the streamlines and pathlines.

(e) Give examples of irrotational and rotational flows.

3. Answer the following questions : ***(any four)*** 5×4=20

(a) Determine the necessary condition that must be satisfied by a given distribution of forces X, Y, Z, so that the fluid may maintain equilibrium.

(b) A quadrant of a circle is just immersed vertically with one edge in the surface, in a liquid, the density of which varies as the depth. Determine the centre of pressure (C.P.).

(c) A box is filled with a heavy gas at a uniform temperature. Prove that if a is the altitude of the highest point above the lowest and p and p' are the pressures at these two points, the ratio of the pressure to the density at any point is equal to

$$\frac{ag}{\log p'/p}.$$

(d) If w is the area of cross-section of a stream filament, prove that the equation of continuity is

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial s}(\rho w q)=0$$

where δs is an element of arc of the filament in the direction of flow and q is the speed.

(e) Determine the acceleration of a fluid particle when velocity distribution is $\vec{a} = \hat{i}\left(Ax^2yt\right) + \hat{j}\left(By^2zt\right) + \hat{k}\left(Czt^2\right)$ where A, B, C are constants. Also find the velocity components.

(f) The velocity field at a point in fluid is given by $\vec{a} = (x/t, y, 0)$. Obtain the pathlines.

4. Answer the following questions : 10×4=40

(a) A mass of homogeneous liquid contained in a vessel revolves uniformly about a vertical axis. You are required to determine the pressure at any point and the surfaces of equal pressure.

OR

A mass m of elastic fluid is rotating about an axis with uniform angular velocity ω, and is acted on by an attraction towards a point in that axis equal to μ times the distance, μ being greater than ω^2. Prove that the equation of a surface of equal density ρ is

$$\mu\left(x^2 + y^2 + z^2\right) - \omega^2\left(x^2 + y^2\right) = k \log\left\{\frac{\mu\left(\mu - \omega^2\right)^2}{8\pi^3} \cdot \frac{m^2}{\rho^2 k^3}\right\}.$$

(b) A hemispherical bowl is filled with water and two vertical planes are drawn through its central radius, cutting off a semi-lune of the surface. If 2α be the angle between the planes, prove that the angle which the resultant pressure on the surface makes with the vertical

$$= \tan^{-1}\left(\frac{\sin\alpha}{\alpha}\right).$$

OR

A gaseous atmosphere in equilibrium is such that $p = k\rho^{\gamma} = R\rho T$ where p, ρ, T are the pressure, density and temperature and k, γ, R are constants. Prove that the temperature decreases upwards at a constant rate α, so that $\frac{dT}{dZ} = -\alpha = -\frac{g}{R} \cdot \frac{\gamma - 1}{\gamma}$. In a certain atmosphere of uniform composition $T = T_0 = \beta z$ where T_0 and β are constants and $\beta < \alpha$. Find the pressure and density and show that they both vanish at height $\frac{T_0}{\beta}$.

(c) Derive the equation of continuity in Cartesian coordinates. Also what happen, if the fluid is homogeneous and incompressible.

OR

Derive the equation of continuity by the Lagrangian method.

(d) The velocity components for a two-dimensional fluid system can be given in Eulerian system by

$$U = 2x + 2y + 3t$$

$$V = x + y + \frac{t}{2}$$

Find the displacement of a fluid particle in the Lagrangian system.

OR

Obtain Euler's equation of motion of a non-viscous fluid in the form $\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla P$.